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APPLICATION OF THE van der POL EQUATION TO OSCILLATOR DESIGN

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U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

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NAVORD REPORT 6771

APPLICATION OF THE van der POL EQUATION TO OSCILLATOR DESIGN

Prepared by:

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ABSTRACT: While the results obtained using the van der Pol equation are generally applicable to all vacuum-tube oscillators, there remains the problem of predicting the behavior of particular oscillator types and determining the effects of the various parameters on operability. To this end the van der Pol equation has been applied to the Hartley, Colpitts, and grounded-grid Colpitts (with autotransformer antenna coupling) oscillator types. Formulas are obtained in terms of the pertinent parameters, for equilibrium amplitude, time constant of amplitude build-up, sensitivity to coherent interference, and noise side-bands. The effect of antenna coupling on these quantities is determined. The results obtained are useful with either grid or plate detection systems.

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U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

This report presents an analysis from a mathematical point of view of those quantities pertinent to vacuum-tube oscillator operability. The material is intended as a guide in the design of oscillators.

The work was performed under authorization of task number NOL-357.

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APPLICATION OF THE van der POL EQUATION TO OSCILLATOR DESIGN

INTRODUCTION

1. Oscillators may be classed in many different ways. Examples of such classifications are to be found in the literature and are characterized by such terms as negative-resistance, feedback, relaxation, etc. Many oscillator types are associated with the name of the inventor, as Hartley, Colpitts, Clapp, Franklin, Meissner, etc. Such classifications have the merit of characterizing the configurations so that they may be distinguished from one-another. However, from a mathematical point of view, it may be said that all oscillators are negative-resistance oscillators. While this unifying viewpoint may not be as familiar as the classifications just cited, it is one to be cultivated since the basis of a more thorough understanding of oscillators depends on adopting just such a viewpoint.

Elementary Considerations

2. The viewpoint just expressed can be made reasonable from elementary considerations. Consider the L-C-R circuit of Figure 1 in shunt with an element ρ whose characteristic we shall determine. It may be readily shown that the differential equation for such a circuit is given by

$$\frac{d^2v}{dt^2} + \frac{1}{C}\left(\frac{1}{\rho} + \frac{1}{R}\right)\frac{dv}{dt} + \frac{v}{LC} = 0 \qquad (1)$$

the solution of equation (1) is given by the relation

$$v = Ae^{-\frac{1}{2L}\left(\frac{1}{R}+\frac{1}{\rho}\right)t} \cos\left[\sqrt{\frac{1}{LC} - \frac{1}{2C}\left(\frac{1}{R}+\frac{1}{\rho}\right)}\,t + \phi\right]. \qquad (2)$$

In order that oscillations build up, it is necessary that the exponent in equation (2) assume a positive value. This can only be so if the quantity $\frac{1}{R} + \frac{1}{\rho}$ is negative, and

$$|\rho| \leq R, \qquad (3)$$

Having by some means or other achieved a value of negative resistance consistent with equation (3), it becomes clear that

while oscillations will build up, they cannot do so indefinitely. Thus the linear differential equation which was used to determine the starting condition gives no information about the equilibrium value of amplitude. However, as is well known, vacuum tube oscillators are capable of limiting the amplitude of oscillation. The implication is then that while a negative resistance is necessary for the start of oscillations, the characteristic supplying the negative resistance can only be negative over a limited region. The limiting action of a vacuum tube oscillator has been adequately explained by van der Pol in reference (a) by suitably modifying equation (1). The modified equation is a second-order non-linear differential equation called van der Pol's Equation. While the results given by van der Pol are generally applicable to all oscillators, there remains the problem of predicting the behavior in particular oscillator types and determining the effect of the various parameters on operability. To this end van der Pol's Equation will be applied to two oscillator types commonly employed; the Hartley and Colpitts types. The circuits for these oscillators are shown in Figure 2.

Negative Resistance in Vacuum Tubes

3. It may be shown that the dynamic characteristic of an oscillator can be determined from the static family of plate characteristics for a triode. Briefly it may be stated, that to the extent that the family of plate characteristics are parallel to one antoher, the plate current may be made dependent on the quantity kV_A, where k is a function of the amplification factor μ and the excitation ratio h such that $k = \mu h - 1$, and V_A is the alternating plate voltage. k is determined by the circuitry external to the tube, and may be positive or negative. For positive values of k the dynamic characteristic takes the form of a cubic equation whose slope at the origin is negative. Figure 3 shows the form of a static characteristic. Temperature-limited emission partly accounts for the upper limit of current. Also shown are several dynamic characteristics, with the excitation ratio as a parameter. The curvature of the characteristic at the top and bottom is such as to cause amplitude limiting.

The Equivalent Circuit

4. Consider the oscillator circuits shown in Figure 2. The equivalent circuits of these oscillators are shown in Figure 4. A sufficient number of elements are shown to emphasize the essential features of operation without destroying the semblance of reality. The antenna impedance is considered to be entirely resistive; the grid circuit, inductors and capacitors are considered lossless, so that

all losses are represented by the antenna resistance. The points G, K and P correspond to the grid, cathode and plate of the triode respectively. The interelectrode capacitance between G and P has been omitted since its effect would be only to alter the operating frequency. The impedance between filament and ground is assumed to be infinite at the frequency of operation. Since we are interested in studying the oscillator under the action of an injected signal, a signal generator is included in the antenna circuit. It is apparent from Figure 4 that the configurations of the oscillators are the same and in order to change from one circuit to another capacitance and inductance interchange their roles.
Referring to Figure 4b, the Colpitts type oscillator, we may write

$$i_{Ra} + i_L + i_2 + i_A = 0 \tag{4}$$

where

$$i_{Ra} = \frac{V - v_i}{Ra}, \tag{4a}$$

$$i_L = \frac{1}{L}\int V\,dt, \tag{4b}$$

$$i_c = C_2 \frac{dV_A}{dt} \tag{4c}$$

The current i_A may be written as a function ϕ of a single variable $V_A + \mu V_G$, where μ is the amplification factor of the triode and V_G is the voltage from grid to cathode. Consequently we may write.

$$i_A = \phi(V_A + \mu V_G) \tag{5}$$

For a Colpitts oscillator the relation between V_G and V_A is given by

$$V_G / V_A = -C_2 / C_1. \tag{6}$$

C_2/C_1 is called the "excitation ratio". A third relation connecting V, V_A and V_G is immediately obvious from Figure 4 and is given by

$$V = V_A - V_G. \tag{7}$$

Substituting for V_G in equation (5) the value given in equation (6) we may write

$$i_A = \phi\left[V_A\left(\mu \frac{C_2}{C_1} - 1\right)\right] = \phi\left(V_A k\right). \tag{8}$$

Substituting equations (4a), (4b), (4c), (6) and (7) into equation (1) and differentiating, the result is given by

$$C_2 \frac{d^2 V_a}{dt^2} + \frac{d i_A}{dt} + \frac{1}{R_a}\left(1 - \frac{C_2}{C_1}\right)\frac{dV_A}{dt} + \frac{V_A}{L}\left(1 - \frac{C_2}{C_1}\right) = \frac{1}{R_a}\frac{dv_i}{dt}. \tag{9}$$

The assumption is made as in reference (a) that the relation between i_A and V_A (negative characteristic - see Figure 3) may be represented by a cubic equation so that

$$\begin{aligned} i_A &= -\alpha V_A + \gamma V_A^3, \\ \alpha &= k\, \partial i_A/\partial V_A \big|_0 \\ \gamma &= \frac{k^3}{3\cdot 2}\, \partial^3 i_A/\partial V_A^3 \big|_0 . \end{aligned} \tag{10}$$

The subscript 0 indicates the derivatives are evaluated at the operating point. Using equation (10), equation (9) becomes

$$\frac{d^2 V_A}{dt^2} - \frac{dV_A}{dt}\cdot\frac{1}{C_{eq}}\left\{\alpha' - G_a' - 3\gamma' V_A^2\right\} + \omega_0^2 V_A = \frac{G_a'}{C_{eq}}\frac{dv_i}{dt} \tag{11}$$

where

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \alpha' = \alpha\frac{C_1}{C_1+C_2}$$

$$\frac{1}{LC_{eq}} = \omega_0^2 \qquad \gamma' = \gamma\frac{C_1}{C_1+C_2}$$

$$G_a = \frac{1}{R_a} \qquad G_a' = \frac{1}{R_a}\frac{C_1}{C_1+C_2}.$$

Equation (11) is a second-order non-linear differential equation called van der Pol's Equation. It is equally applicable to the Hartley oscillator if the following relations are employed:

$$\frac{1}{C} = \frac{1}{C_{eq}} \qquad \alpha' = \alpha \frac{n_A}{n_A + n_G}$$

$$\frac{1}{LC} = \omega_0^2 \qquad \gamma' = \gamma \frac{n_A}{n_A + n_G}$$

$$G_a = \frac{1}{R_a} \qquad G_a' = \frac{1}{R_A} \frac{n_A}{n_A + n_G}.$$

The limiting effects are accounted for in equation (11) by a damping term proportional to V_A^2. The normal form of van der Pol's Equation is generally given by

$$\frac{d^2 u}{dT^2} - \varepsilon(1-u^2)\frac{du}{dt} + u = Q'\frac{ds}{dt}, \text{ where}$$

$$\varepsilon = \frac{\alpha' - G_a}{\omega_0 C_{eq}}, \qquad Q' = \frac{G_a'}{C_{eq}\omega_0}, \tag{12}$$

$$u = V_A \Big/ \sqrt{\frac{\alpha' - G_a}{3\gamma'}}, \quad T = \omega_0 t, \quad S = v_i \Big/ \sqrt{\frac{\alpha' - G_a}{3\gamma'}}.$$

The coefficient ε determines the nature of the oscillations. For $\varepsilon < 2$ the oscillations are essentially harmonic, for $\varepsilon > 2$ the oscillations depart markedly from a sine wave and are characterized by the term "relaxation oscillations".

Oscillation Build-Up

5. Consider equation (11) with the forcing function set equal to zero. With the aid of the solution for equation (12) given in reference (a) the solution for equation (11) is given by the relation

$$V_A = \sqrt{\frac{4}{3}\frac{\alpha' - G_a}{\gamma'}} \cdot \frac{\cos(\omega_0 t + \phi_0)}{\sqrt{1 + e^{-(t-t_0)(\alpha' - G_a)/C_{eq}}}}, \tag{13}$$

where $\sqrt{\frac{4}{3}\frac{\alpha' - G_a}{\gamma'}} = V_0$

is the amplitude of the free oscillation as $t \to \infty$. ϕ_0 takes account of the oscillation phase at time t_0. In arriving at equation (13) it was assumed that

$$\frac{dV_A(t)}{dt} << V_A(t), \text{ and} \tag{14}$$

$$\varepsilon < 1.$$

The quantity $\frac{\alpha' - G_a}{C_{eq}}$ in equation (13) is the reciprocal of the time constant τ associated with amplitude build-up. Using the parameters of interest an expression is obtained for τ such that

$$\frac{1}{\tau} = \frac{\left(\mu \frac{C_2}{C_1} - 1\right)\left(\frac{\partial i_A}{\partial V_A}\right)_0 \frac{C_1}{C_1 + C_2} - G_a}{C_1 C_2 / C_1 + C_2} \quad . \tag{15}$$

The condition that oscillations start is the presence of a negative resistance. Comparing equation (11) with equation (2) a negative resistance is initially present if $\alpha' > G_a$.

Solution with Coherent Forcing Function

6. Consider now the case where the forcing function v_i' is related to V_A. Such a situation is easily realizable physically. In such cases the forcing function is indistinguishable from a damping term so that equation (11) may be written in the form

$$\frac{d^2 V_A}{dt^2} - \frac{dV_A}{dt} \cdot \frac{1}{C_{eq}} \left\{ \alpha' - G_a - m G_a' \cos\psi \right\} + \frac{3\gamma'}{C_{eq}} V^2 \frac{dV_A}{dt} + \omega_0^2 V_A = 0 \tag{16}$$

Following the procedure used in obtaining equation (13), the modified solution in the steady state is

$$V_A = \sqrt{4/3 \frac{(\alpha' - G_a - m G_a' \cos\psi)}{\gamma'}} \cos(\omega_0 t + \phi_0) \tag{17}$$

If the quantity $\frac{m G_a' \cos\psi}{\alpha' - G_a}$ is small with respect to unity equation (17) may be put into the form

$$V_A = V_0 \left(1 - \frac{1}{2} \frac{m G_a'}{\alpha' - G_a} \cdot \cos\psi\right) \cos(\omega_0 t + \phi_0), \quad \text{where} \quad V_0 = \sqrt{4/3 \frac{(\alpha' - G_a)}{\gamma'}} \; . \tag{18}$$

The result is an amplitude modulated wave. ψ is obviously the phase angle between outgoing and return wave, so that

$$\psi = 2\pi \cdot \frac{2H}{\lambda} + \Omega, \tag{19}$$

where H is the distance traveled by the wave before its return from a reflector, to the oscillator. Differentrating equation (19) with respect to time we have

$$\frac{1}{2\pi}\frac{d\psi}{dt} = f = 2\,\frac{dH/dt}{\lambda} = \frac{2s}{\lambda} \qquad (20)$$

where s is the relative velocity between oscillator and reflector, and f is the frequency of modulation. The modulation factor m may be given by

$$m = p/2H, \qquad (21)$$

where p is a constant depending on the radiator associated with the oscillator. The change in RF voltage may be put in the form

$$|V_A - V_0| = -\frac{1}{2}\left(\frac{V_0\,G_a'}{\alpha' - G_a}\right)\frac{p}{2H}\cos\left(\frac{4\pi H}{\lambda} + \Omega\right). \qquad (22)$$

Equation (22) may be put in the more compact form

$$\frac{\Delta V}{V_0} = Re\,\frac{A}{H}\,e^{i(4\pi H/\lambda + \Omega)}. \qquad (23)$$

The sensitivity S of an oscillator may be defined in terms of the relative change in output voltage with respect to the relative change in conductance such that

$$S = \frac{\Delta V_0}{V_0} \Big/ \frac{G_a}{\Delta G_a} \qquad * \qquad (24)$$

*Consider a horizontal half-wave antenna at a height H above a perfectly conducting plane. By the "Method of Images", the resultant field at the antenna due to the image antenna is $E_\theta = i\,30/2H\,I_0 e^{i\omega t}\cdot e^{i(4\pi H/\lambda - \pi)}$. The voltage induced in the antenna by the image antenna may be obtained by multiplying the field intensity at the antenna by the equivalent length $l_e = \lambda/\pi$ of the antenna. Thus the induced voltage $V_{12} = \frac{\lambda}{\pi}\frac{30}{2H} I_0 e^{i\omega t} e^{i(4\pi H/\lambda + \frac{3\pi}{2})}$. The mutual impedance between the antenna and its image is therefore $Z_{12} = V_{12}/I_0 e^{i\omega t} = 30\lambda/2\pi H\cdot e^{i(4\pi H/\lambda + \Omega')}$ Dividing both sides of this equation by the resistance of a half-wave antenna $30\,Cin\,2\pi$, we readily obtain $G_a/\Delta G_a = Re\,B/H\,e^{i(4\pi H/\lambda + \Omega)}$. Sensitivity as defined by equation (24) gives $S = \frac{A}{B}$.

Using the steady state amplitude given by equation (13) we find that

$$S = -\frac{1}{2}\left(\frac{G_a}{\alpha' - G_a}\right) \tag{25}$$

It is apparent from equation (25) that the sensitivity is greatest when α' is equal to G_a, i.e. when the oscillator is on the verge of oscillation.

Noise Considerations

7. When an oscillator is used for the purpose of generating a signal ultimately to be radiated, the coupling between the antenna and the tank circuit of the oscillator is of some importance. In the case of interest, we shall consider the antenna coupling device to be an autotransformer. The matching condition for the antenna may be found by altering the antenna tap position of the tank coil. The transferred antenna resistance will obviously be a function of the coupling. Two noise sources are to be considered with respect to antennas, namely thermal noise due to ohmic resistance and antenna noise associated with radiation resistance. The distinction between the two sources lies in the fact that the temperature associated with radiation resistance is effective temperature which describes the noisiness of the antenna location. The effective temperature is in general higher than the ambient temperature. The remainder of the oscillator has associated with it noise sources arising from the tube (shot noise) and circuit noise (thermal noise). In the interest of simplification, these additional sources will not be included in this study.

Equivalent Circuit of Oscillator With Autotransformer Coupling

8. To study the effect of noise sources, we choose as our oscillator a grounded-grid Colpitts type, with the antenna coupled to the tank circuit by means of an autotransformer. The equivalent circuit is shown in Figure 5. Appendix A shows the derivation of the differential equations which may be put in a form suitable for studying the voltage V_A as previously, or for studying the voltage V_g. The equation for V_g with a noise forcing function (see Appendix A) is given by

$$\frac{d^2 V_g}{dt^2} - \frac{1}{C_{eq}}\left\{h(\alpha' - G_a) - \frac{3\gamma'}{h} V_g^2\right\}\frac{dV_g}{dt} + \omega_0^2 V_g = \frac{h}{C_{eq}}\frac{d}{dt}\left\{\sqrt{\overline{v_N^2}}\, G_a' h\right\}, \tag{27}$$

where the noise sources have been added in the manner necessary to give the total quadratic content. The coherent forcing function may be neglected here, or else its effect may be included with G_a. The combined noise source constitutes a non-coherent interference source and can not be treated in the same manner as a coherent interference source.

Noise Signal in Narrow Transmission Systems

9. The action of the noise source shown in equation (27) is independent of its origin. For the time being nothing will be said about the explicit form of the noise voltages. The total quadratic content will be denoted by the relation

$$\overline{I}_N^2 = \overline{v}_N^2 (G_a' k)^2 \tag{28}$$

It is shown in reference (b) that when noise passes thru a narrow radio-frequency transmission system the character of the noise changes. The noise has the appearance of a carrier wave with random amplitude variation. The total noise signal may then be expressed by the relation,

$$i_N = R_N \cos(\omega_1 t + \theta). \tag{29}$$

ω_1 is the carrier frequency, which is chosen within the pass band, and R_N is the amplitude of the envelope, which has a two-dimensional normal distribution and may be expressed by

$$P(R_N)\,dR_N = 2A^2 R_N e^{-A^2 R_N^2}\,dR_N$$

where

$$\overline{R}_N^2 = \frac{1}{A^2} = 2\overline{I}_N^2 \tag{30}$$

Values of θ are equally probable. Figure 6 shows the appearance of noise after having passed through a narrow band transmission system. We shall take as our noise voltage a sine wave whose amplitude is the rms-value of the distribution given by equation (30) and zero phase. The frequency of the sine wave lies within the band-pass.

Differential Equation with Noise Source

10. Employing the results of the previous section we may write van der Pol's Equation in the form

$$\frac{d^2V_g}{dt^2}-\frac{1}{C_{ag}}\left\{h(\alpha'-G_a)-3\gamma' V_g^2\right\}\frac{dV_g}{dt}+\omega_0^2 V_g=\frac{h}{C_{ag}}\sqrt{2\bar{I}_N^2}\cdot\omega_1\cos\omega_1 t \quad (31)$$

where ω_1 lies within the narrow band in the neighborhood of ω_0. The coherent signal has been omitted for simplicity. Equation (31) may be written more compactly using the following notation,

$$\frac{d^2V_g}{dt^2}-\left\{\alpha''-3\gamma' V_g^2\right\}\frac{dV_g}{dt}+\omega_0^2 V_g=\omega_1 I\cos\omega_1 t \quad (32)$$

We have here van der Pol's Equation with a forcing function. In general there are two regions of interest which have been discussed by van der Pol in solving equation (32); (a) when the forcing function and the oscillator frequency are the same, or region of "lock-in" and (b) when both signals are present, or "region of beats".

(a) Lock-In Region

When ω_1 is close to ω_0 a solution of the form

$$V_g = b_1 \sin\omega_1 t + b_2\cos\omega_1 t \quad (33)$$

is assumed, where b_1 and b_2 are slowly varying functions of time. Retaining only fundamental frequency terms, van der Pol has derived a set of differential equations describing the action of the system. They are

$$2\frac{db_1}{dt}+z b_2-\alpha'' b_1\left(1-\frac{b^2}{a_0^2}\right)=0$$
$$2\frac{db_2}{dt}+z b_1-\alpha'' b_1\left(1-\frac{b^2}{a_0^2}\right)=I. \quad (34)$$

where

$$z=\frac{\omega_0^2-\omega_1^2}{\omega_1}\doteq 2(\omega_0-\omega_1)$$

a_0 = amplitude of free oscillation, and

$b_2 = b_1^2+b_2^2$.

A particular solution of these equations is obtained when db_1/dt and db_2/dt are set equal to zero, so that

$$b^2\left\{z^2+\alpha''^2\left(1-\frac{b^2}{a_0^2}\right)\right\}=I^2. \quad (35)$$

There are in general three different solutions to equation (35). However, if I is small, then b^2/a_0^2 is nearly unity so that

$$z^2 = \frac{I^2}{b^2} \doteq \frac{I^2}{a_0^2} = \frac{I^2}{V_{g0}^2} \tag{36}$$

Since the noise signal I is generally much smaller than the amplitude of free oscillation, the lock-in region is very small.

(b) Region of Beats

We now consider the noise sidebands in the region of beats. Here van der Pol has assumed a solution of the form which contains both free and forced oscillations such that

$$v = a \sin(\omega_0 t + \delta) + b \sin(\omega_1 t + \lambda) \tag{37}$$

Proceeding as in the previous case van der Pol has shown that the solution here is given by the relations

$$\omega - \omega_0 = 0 \qquad \text{(a)}$$

$$a\left(1 - \frac{a^2 + 2b^2}{a_0^2}\right) = 0 \qquad \text{(b)} \qquad (38)$$

$$b^2\left\{z^2 + \alpha''^2\left(1 - \frac{b^2 + 2a^2}{a_0^2}\right)^2\right\} = I^2 \qquad \text{(c)}$$

From equation (38)(a) we see the presence of the free oscillation. From equation (38)(b) we see two solutions,

$$a = 0, \qquad \text{(a)}$$

$$(39)$$

$$1 - \frac{a^2 + 2b^2}{a_0^2} = 0 \qquad \text{(b)}$$

When $a = 0$ then the system is in such a state that only one frequency is present as given by the lock-in solution. Equation (39)(b) shows that both free and forced oscillations are present. Substituting from equation (39)(b) into equation (38)(c) for a^2 we have

$$b^2\left\{z^2+\alpha''^2\left(1-\frac{3b^2}{a_0^2}\right)^2\right\}=I^2 \tag{40}$$

If the forced oscillations are small i.e., $b^2 << a_0^2$ then we may write

$$b^2=\frac{I^2}{z^2+\alpha''^2}, \tag{41}$$

which for values of z small with respect to α'' show that the noise side bands are constant. Thus

$$b^2=\frac{I^2}{\alpha''^2}=\frac{2\left(\bar{v}_{NO}^2+\bar{v}_{NA}^2\right)(G_a' h)^2}{(\alpha'-G_a)^2}, \tag{42}$$

The result given by equation (42) is in accord with results found in reference (c). The total noise content $\bar{v}_N^2$ of the antenna due to the ohmic and radiation resistance components is given in reference (b) by the famous Nyquist relation

$$\bar{v}_N^2=\bar{v}_{NO}^2+\bar{v}_{NA}^2=4K\,df\,(TR_O+T_E R_A) \tag{43}$$

where $\bar{v}_{NO}^2$ = the noise content of the ohmic component of antenna resistance R_O,

$\bar{v}_{NA}^2$ = the noise content of the radiation resistance component of antenna resistance R_a,

K = Boltzmann's constant,

df = bandwidth of measuring system,

T = ambient temperature in °Abs., and

T_E = effective temperature of antenna location in °Abs.

Signal-to Noise-Ratio

11. It is apparent from equation (43) that a high Q circuit (small df) will reduce the noise side bands as will the introduction of low-noise tubes, other things being equal. It is of interest to note the total conductance $(\alpha'-G_a)$ is a quantity pertinent to all the formulas developed. Consider

now the expression given by equation (22), showing the change in oscillator voltage during coherent interference, and the mean square value of the noise side-bands in the neighborhood of the signal as given by equation (42). In order that equation (22) apply at the grid it should be multiplied by the excitation ratio h. Using rms values we have for the ratio of these two quantities δ,

$$\delta = \frac{\Delta V_g}{\sqrt{\overline{e^2}}} = \frac{1}{2}\,\frac{V_0 G_a'}{\sqrt{\overline{I_N^2}}}\cdot\frac{\rho h}{2H} \tag{44}$$

which we simplify immediately to

$$\delta = k\, G_a \sqrt{\alpha' - G_a} \tag{45}$$

since V_0 is proportional to $\sqrt{\alpha' - G_a}$. δ maximizes so that

$$G_a = \frac{2}{3}\alpha'. \tag{46}$$

SUMMARY

13. Several oscillator types have been analyzed using van der Pol's Equation. Information pertinent to the understanding of such quantities as time-constant of oscillation build-up, equilibrium amplitude, coherent interference and noise side-bands has been derived. The method and results presented should be of use in the design of oscillators.

APPENDIX A

DERIVATION OF Van der Pol's EQUATION FOR THE CIRCUIT OF FIGURE (5)

Referring to Figure (5) we may write

(a) $i_R = \dfrac{v - v_i'}{R_A + R_0}$ (d) $i_{C_2} = C_2 \dfrac{dV}{dt}$

(b) $i_{L_1} = \dfrac{1}{L_1}\int v\,dt$ (e) $i_a = f(V_A)$

(c) $i_{L_2} = \dfrac{1}{L_2}\int v'\,dt$ (f) $\dfrac{V_g}{V_A} = -\dfrac{C_2}{C_1} = h$

(A-1) (g) $V_A - V_g = v + v'$

(h) $\dfrac{v+v'}{v} = \varkappa = \dfrac{\mathcal{L}}{L_1+M} = \dfrac{L_1+L_2+2M}{L_1+M}$

Coefficient of coupling assumed unity

(i) $\dfrac{v'}{v} = \varkappa - 1$

Using Kirchoff's current law we have

(a) $i_a + i_{C2} + i_{L2} = 0$

(A-2)

(b) $i_R + i_{L_1} - i_{L_2} = 0$

Eliminating i_{L_2} we have

(A-3) $i_a + i_{C_2} + i_{L_1} + i_R = 0.$

Substituting from (A-1) into (A-3) we have

$$\frac{v}{R_A+R_0}+\frac{1}{L_1}\int v\,dt+C_2\frac{dV_A}{dt}+f(V_A)=\frac{v_i}{R_A+R_0} \tag{A-4}$$

Substituting for v from (A-1g) and V_g from (A-1f) we may write

$$\frac{V_A}{R_A+R_0}+\frac{h\,V_A}{R_A+R_0}-\frac{v'}{R_A+R_0}+\frac{1}{L_1}\int V_A\,dt+\frac{h}{L_1}\int V_A\,dt$$
$$-\frac{1}{L}\int v'\,dt+C_2\frac{dV_A}{dt}+f(V_A)=\frac{v_i}{R_A+R_0} \tag{A-5}$$

Using the relations obtained from (A-1), we may write

$$v'=\frac{(1+h)(K-1)}{K}\,V_A, \tag{A-6}$$

which permits equation (A-5) to be in the form, after differentation and simplification

$$\frac{d^2V_A}{dt^2}+\frac{1}{K(R_A+R_0)}\cdot\frac{1}{C_{eq}}\frac{dV_A}{dt}+\frac{V_A}{KL_1C_{eq}}+\frac{f'(V_A)}{C_2}=\frac{1}{R_A+R_0}\cdot\frac{1}{C_2}\frac{dv_i}{dt}. \tag{A-7}$$

If we assume as in the previous cases that the vacuum tube characteristic is cubic, such that

$$i_A=-\alpha V_A+\gamma V_3^3. \tag{A-8}$$

then equation (A-7) becomes

$$\frac{d^2V_A}{dt^2}+\left[\frac{1}{K(R_A+R_0)}\cdot\frac{1}{C_{eq}}-\frac{\alpha}{C_2}+\frac{3\gamma}{C_2}V_A^2\right]\frac{dV_A}{dt}$$
$$+\frac{V_A}{KL_1C_{eq}}=\frac{1}{C_2}\cdot\frac{1}{R_A+R_0}\cdot\frac{dv_i}{dt}. \tag{A-9}$$

After simplification (A-9) becomes

$$\frac{d^2V_A}{dt^2}-\frac{1}{C_{eq}}\left\{(\alpha'-G_a)-3\gamma'V_A^2\right\}\frac{dV_A}{dt}+\omega_0^2V_A=\frac{K}{C_{eq}}G_a'\frac{dv_i}{dt} \tag{A-10}$$

where

$$\frac{1}{C_{eq}} = \frac{1}{C_1C_2/C_1+C_2}, \qquad G_a = \frac{1}{K(R_A+R_0)},$$

$$\alpha' = \frac{C_1}{C_1+C_2}\alpha, \qquad G_a' = G_a\frac{C_1}{C_1+C_2}.$$

$$\gamma' = \frac{C_1}{C_1+C_2}\gamma,$$

Since we shall prefer to work with the grid voltage Vg. we may transform (A-10) using (A-1f) so that

$$\text{(A-11)} \quad \frac{d^2V_g}{dt^2} - \frac{1}{C_{eq}}\left\{h(\alpha'-G_a) - \frac{3\gamma'}{h}V_g^2\right\}\frac{dV_g}{dt} + \omega_0^2V_g = \frac{hk}{C_{eq}}G_a'\frac{dV_A}{dt}$$

If a noise source is added in the antenna circuit then we must add a term $\frac{hk}{C_{eq}}G_a'\frac{d}{dt}\sqrt{\overline{v_N^2}}$ where $\overline{v_N^2}$ is the quadratic content of the antenna noise and $v_N^2 = \overline{v_0^2} + \overline{v_A^2}$.

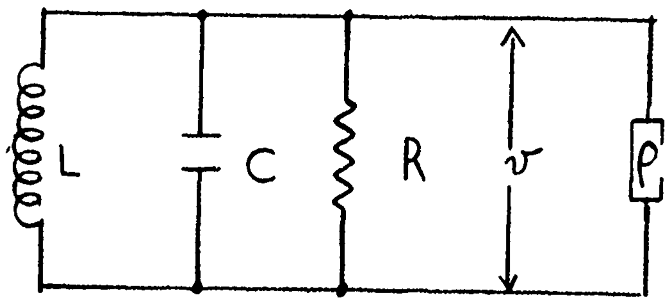

Figure 1

Oscillator with Negative Resistance Element

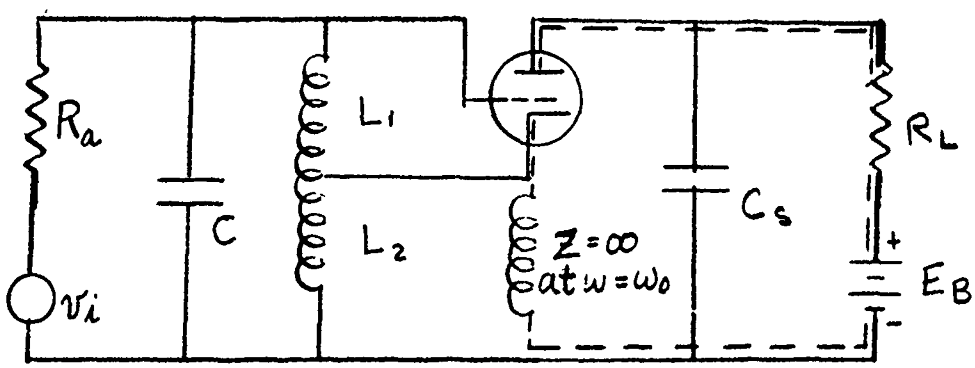

Figure 2a
Circuit Diagram of Hartley Oscillator

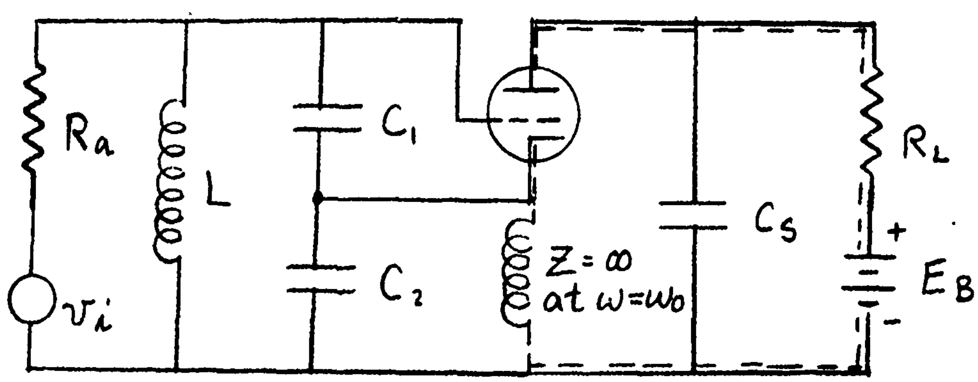

Figure 2b

Circuit Diagram of Colpitts Oscillator

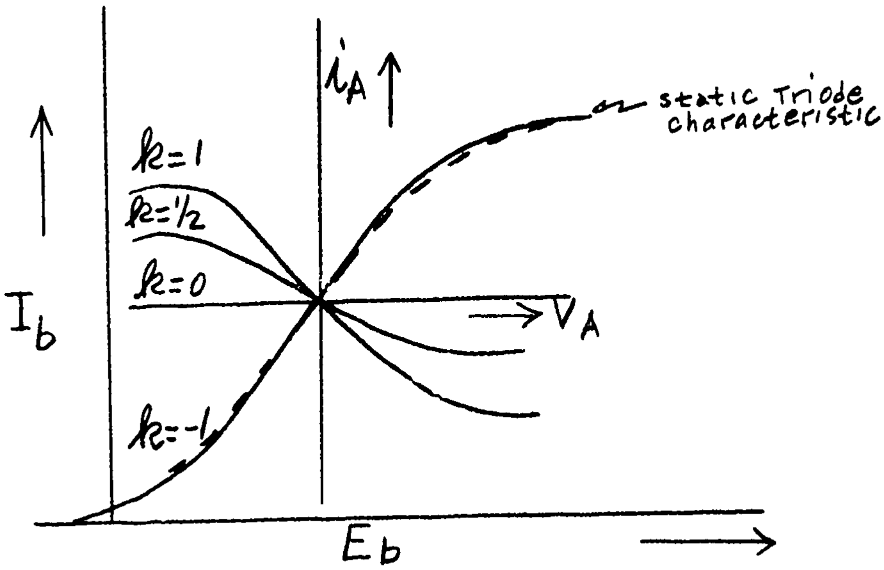

Figure 3
Triode Oscillator Characteristics

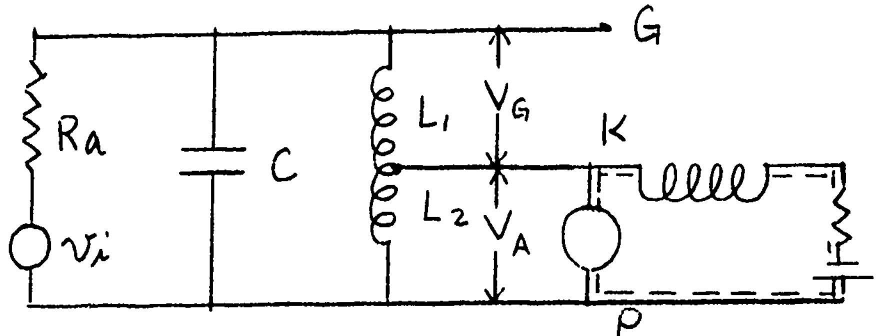

Figure 4a
Equivalent Circuit of Hartley Oscillator

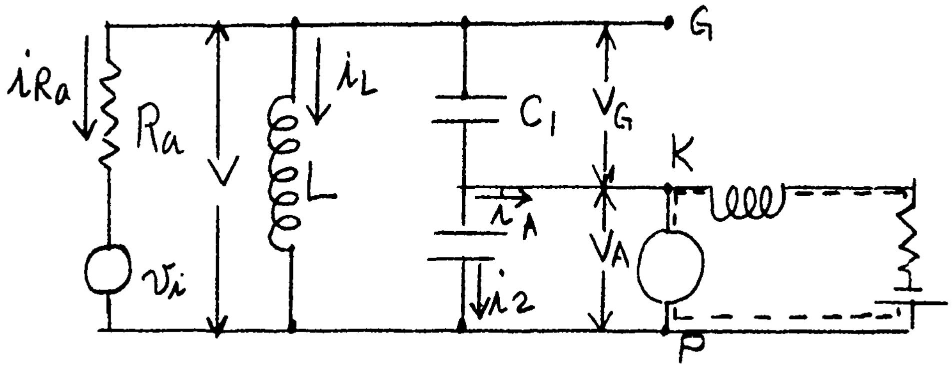

Figure 4b
Equivalent Circuit of Colpitts Oscillator

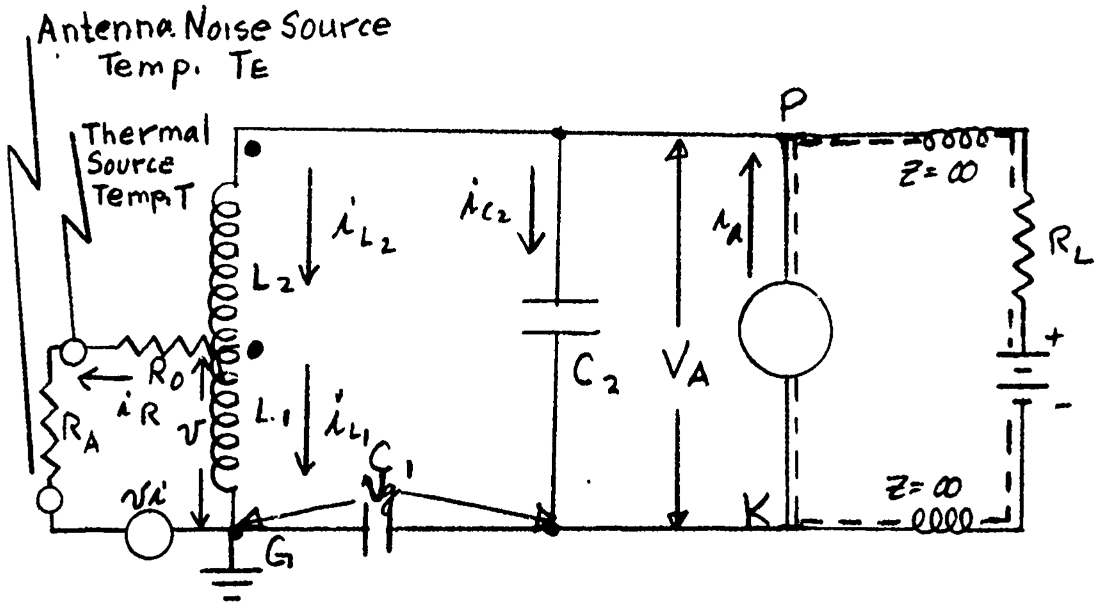

Figure 5

Equivalent Circuit of Grounded Grid Colpitts Oscillator with Autotransformer Antenna Coupling

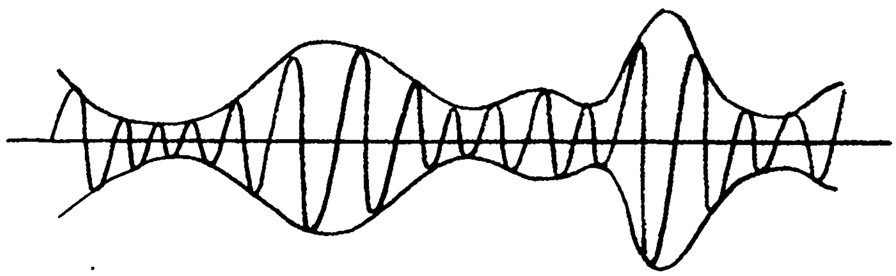

Figure 6

Appearance of Narrow Band Noise

References

(a) Van der Pol, B., The Non-linear Theory of Electric Oscillations, Proc. I. R. E., 22, 1934.

(b) Goldman, S., Frequency Analysis, Modulation and Noise, McGraw-Hill Book Co., New York, 1948.

(c) Van der Ziel, A., Noise, Prentice-Hall, Inc., New York, 1954.

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